

Local gauge theory and coarse graining

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Abstract. Within the discrete gauge theory which is the basis of spin foam models, the problem of macroscopically faithful coarse graining is studied. Macroscopic data is identified; it contains the holonomy evaluation along a discrete set of loops and the homotopy classes of certain maps. When two configurations share this data they are related by a local deformation. The interpretation is that such configurations differ by “microscopic details”. In many cases the homotopy type of the relevant maps is trivial for every connection; two important cases in which the homotopy data is composed by a set of integer numbers are: (i) a two dimensional base manifold and structure group $U(1)$, (ii) a four dimensional base manifold and structure group $SU(2)$. These cases are relevant for spin foam models of two dimensional gravity and four dimensional gravity respectively. This result suggests that if spin foam models for two-dimensional and four-dimensional gravity are modified to include all the relevant macroscopic degrees of freedom –the complete collection of macroscopic variables necessary to ensure faithful coarse graining–, then they could provide appropriate effective theories at a given scale.

1. Connections (modulo gauge) in terms of local holonomy variables

Let (E, π, M) be a principal G -bundle over a d -dimensional base space M and \mathcal{A}_π be the space of connections on it.

In order to introduce a concept of measuring scale and locality we use a smooth triangulation Δ of M ; alternatively, we can use a cellular decomposition that refines a smooth triangulation. This decomposition is inherited by the bundle (Figure 1.a).

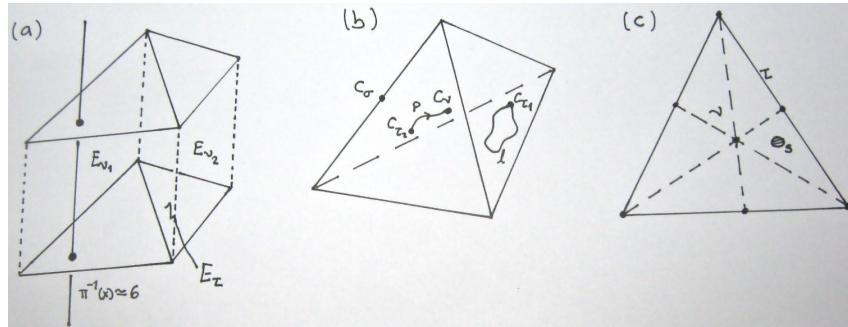


Figure 1. (a) The smooth G -bundles π_{ν_1} and π_{ν_2} intersect at the smooth G -bundle π_τ .
 (b) Paths and loops in $\mathcal{P}(\nu)$.
 (c) $\text{Sd}(\nu)$, the baricentric subdivision of ν . We also show s , the support of B in the example.

Definition 1 *Decomposition of the bundle (see figure 1.a):*

- The principal G -bundle over M , $(E_\Delta, \pi_\Delta, M_\Delta)$, is called Δ -smooth if its restriction to $E_\nu = \pi_\Delta^{-1}(\nu)$ for each simplex ν of the triangulation is the restriction of a smooth bundle to an open neighborhood of $E_\nu \subset E_\Delta$.
- $\mathcal{A}_{\pi_\tau}^\infty$ is the space of smooth connections in the bundle $\pi_\tau = \pi|_{\tau \in (|\Delta|, \phi)}$ (restrictions to π_τ of connections which are smooth in an open neighborhood of the subbundle π_τ of π , where τ is a simplex in the triangulation).
- $\bar{\times}_{\tau \in (|\Delta|, \phi)} \mathcal{A}_{\pi_\tau}^\infty$ is the subset of the cartesian product defined by the compatibility condition $\sigma \subset \tau \Rightarrow A_\sigma = A_\tau|_\sigma$ for every two simplices of the triangulation. We will write $\mathcal{A}_\pi^{\Delta-\infty} = \bar{\times}_{\tau \in (|\Delta|, \phi)} \mathcal{A}_{\pi_\tau}^\infty$.
- A Δ -smooth bundle map is a homeomorphism of the total space $\tilde{f} : E \rightarrow E$ which preserves each subbundle π_τ and is the restriction to π_τ of a smooth bundle map on an open neighborhood of it. Clearly a Δ -smooth bundle map acts on connections by pull back and sends $\mathcal{A}_\pi^{\Delta-\infty}$ to itself.
- $\mathcal{G}_{\star, \pi}^{\Delta-\infty}$ is the group of Δ -smooth bundle equivalence maps which consists of Δ -smooth bundle maps which induce the identity map on the base space M . We will write $\mathcal{A}/\mathcal{G}_{\star, \pi}^{\Delta-\infty} = \mathcal{A}_\pi^{\Delta-\infty}/\mathcal{G}_{\star, \pi}^{\Delta-\infty}$.

Now we present local holonomy data for the bundle and the connection. It is the application of a construction of Barrett and Kobayashi to the subbundles over the simplices.

In simplex ν of the triangulation choose its baricenter $C\nu$ as base point, and consider the space $\mathcal{L}(C\nu, \nu)$ of piecewise smooth oriented $C\nu$ -based loops modulo reparametrization and modulo retracing. It turns out that $\mathcal{L}(C\nu, \nu)$ is a group. Once we have identified G with $\pi^{-1}(C\nu)$ the holonomy of a given connection $A \in \mathcal{A}_{\pi_\nu}^\infty$ around a loop $l \in \mathcal{L}(C\nu, \nu)$ is the group element $\text{Hol}(l, A) \in G$. Moreover, if we fix the connection the holonomy map gives us a group homomorphism

$$\text{Hol}_A \equiv \text{Hol}(\cdot, A) : \mathcal{L}(C\nu, \nu) \rightarrow G.$$

Gauge transformations act on holonomies only at the base point and the action is by conjugation.

Holonomy evaluations are specified by points of the space $\mathcal{A}/\mathcal{G}_{\star, \pi_\nu}$, the space of connections modulo the group of gauge transformations whose restriction to the fiber $\pi^{-1}(C\nu)$ is the identity. Holonomy evaluations can be used to construct a trivialization of the bundle over ν , and they characterize the connection (modulo gauge); this is an application of a theorem of Barrett and Kobayashi [1] to the simple case where the base space is a topological disc.

The theorem provides a construction of the bundle and the connection. Here is a sketch of the main idea. The total space is given by

$$E_\nu = G \times \mathcal{P}_{C_\nu} / \sim_{\text{Hol}^\nu}$$

where \mathcal{P}_{C_ν} is the space of paths in ν (modulo reparametrization and retracing) whose source is C_ν . The idea behind the equivalence relation is that a pair $(g, p) \in G \times \mathcal{P}_{C_\nu}$ gives a point on the fiber over $t(p)$, the target of p ; g specifies a point on the fiber at the beginning of the path and it is parallel transported to the end of the path. One only has to acknowledge that (g_1, p_1) and (g_2, p_2) determine the same point if $t(p_1) = t(p_2)$ and the parallel transports agree, $\text{PT}_{p_1} g_1 = \text{PT}_{p_2} g_2$. This last condition can also be written as $g_1 H^\nu(p_2^{-1} \circ p_1) = g_2$.

If this construction is applied to each simplex of the smooth triangulation, the connection and the bundle will be described by means of holonomies. However, a collection of such holonomy evaluations for each simplex will be related to a bundle and a connection on it only if certain compatibility conditions hold. The pasting of descriptions over the different simplices needs

extra data. The extra data glues the bundle over a simplex to the bundle over any simplex that intersect it.

Consider two simplices of the smooth triangulation such that $\tau \subset \nu$. Thus, for each point $x \in \tau$ the above construction gives two constructions of π_x : $E_\tau|_x$ and $E_\nu|_x$. The glue for these bundles is again parallel transport. There are two groups $\mathcal{L}(C\tau, \tau)$ and $\mathcal{L}(C\nu, \nu)$; and there are also paths in ν between $C\tau$ and $C\nu$. The three spaces can be considered together forming a semigroup; two of its elements can be composed only if the target of the first is the same as the source of the second. Consider all the subsimplices of ν , $\tau_i \subset \nu$. The semigroup of paths in ν which have sources and targets at $C\nu$ or any of the C_{τ_i} will be called $\mathcal{P}(\nu)$ (see figure 1.b). Notice that this semigroup contains $\mathcal{L}(C\nu, \nu)$ and $\mathcal{L}(C\tau_i, \tau_i)$ for each $\tau_i \subset \nu$. Once the fibers over $C\nu$ and all the C_{τ_i} s have been identified with G , parallel transport is described by the semigroup homomorphism

$$\text{PT}^\nu : \mathcal{P}(\nu) \rightarrow G.$$

PT^ν contains all the information in H^ν and each of the H^{τ_i} ; in addition, it induces a further equivalence relation that glues the bundle over ν with the bundles over each τ_i . For example, $(g_1, p_1) \in E_\nu|_x$ is equivalent to $(g_2, p_2) \in E_\tau|_x$ if $g_1 \text{PT}^\nu(p_2^{-1} \circ p_1) = g_2$.

We can extend this construction to glue the bundles over all the simplices of the smooth triangulation to obtain a variation of the Barrett and Kobayashi theorem which is tailored to the sense of locality given by the triangulation. The ingredients are $\mathcal{P}(\Delta)$ –the semigroup associated to the smooth triangulation of the manifold- and a parallel transport map which will be called simply PT . $\mathcal{P}(\Delta)$ is generated by composing paths (modulo reparameterization and retracing) contained in some simplex of the triangulation and with source and target being baricenters of simplices of the triangulation. PT is required to be a Δ -smooth semigroup homomorphism. The smoothness condition means that its restriction to paths contained in any simplex is the restriction of a map on paths contained in an open neighborhood of the simplex which is smooth in the sense of Barrett [1]. A detailed study of the abelian case was presented at [2].

Theorem 1 *Let M be a manifold, Δ a smooth triangulation of it, and let $\text{PT} : \mathcal{P}(\Delta) \rightarrow G$ be a Δ -smooth semigroup homomorphism. Then, there is a Δ -smooth G -bundle (E, π_Δ, M) , a collection of points $\{b_\nu \in \pi^{-1}(C\nu)\}_{\nu \in \Delta}$, and a connection $A \in \mathcal{A}_{\pi_\Delta}$ such that $\text{PT} = \text{PT}_A$. The bundle and the connection are unique up to a bundle equivalence transformation.*

2. Macroscopic variables and faithful coarse graining

The last theorem describes the bundle and the connection in terms of a parallel transport map.

Recall that since bundles over discs are trivial and simplices have the topology of the disc. Different parallel transport maps may yield inequivalent bundles only because they glue *standard building blocks* in inequivalent ways determined by the parallel transport data.

Each $\mathcal{P}(\nu)$ is decomposed into subsemigroups that intersect each other as dictated by the simplicial structure of triangulation

$$\nu_1 \cap \nu_2 = \tau \Rightarrow \mathcal{P}(\nu_1) \cap \mathcal{P}(\nu_2) = \mathcal{P}(\tau).$$

This property lets us treat one simplex at a time, and see non trivial gluing as caused by the gluing of individual bundles over simplices with the bundles over their boundary faces.

If $\tau \subset \partial\nu$ then Theorem 1 provides π_ν , π_τ and a gluing map $I_{\tau\nu} : \pi_\tau \rightarrow \pi_\nu$ determined by PT^ν . We will show that there is macroscopic data that can be extracted from PT^ν assuring that if $\text{Data}_\Delta(\text{PT}_1^\nu) = \text{Data}_\Delta(\text{PT}_2^\nu)$ then $I_{\tau\nu}$ is equivalent to $I_{\tau\nu}$. A realization of this goal in the abelian case was presented at [2].

Below is an example of inequivalent gluings. It shows that $\text{Data}_\Delta(\text{PT}^\nu)$ must include some extra data apart from a set of parallel transport evaluations along a discrete collection of paths and loops.

Example:

Let $G = U(1)$ and $M \sim S^2$ with Δ a smooth triangulation of it; also let $\nu \in \Delta$ be a triangle and $\tau \in \Delta$ be one of its one dimensional faces. The baricentric subdivision of ν , $Sd(\nu)$, is shown in figure 1.c.

In S^2 we have a parallel transport map PT_1 such that PT_1^ν is a flat parallel transport (all holonomies are the identity). In an appropriate local trivialization over ν , PT_1^ν would be described by a connection one form $A_1 = 0$. We have a second parallel transport map PT_2 such that for every $\delta \in \Delta$ such that $\delta \neq \nu$ we have $PT_2^\delta = PT_1^\delta$.

Let us use the trivialization over ν according to which $A_1 = 0$. PT_2^ν described in the same trivialization is characterized by a connection one form A_2 determined by a curvature two form B_2 which is zero everywhere except for the compact support $s \subset \nu$ shown in figure 1.c. The size of $B_2|_s$ is adjusted in such a way that $\int_D B_2 = 2\pi$, where D is the triangle in $Sd(\nu)$ containing s .

All the gluings between bundles over simplices determined by PT_2 agree with those determined by PT_1 except for the gluing of the bundles over ν and τ and ν and ν' (the triangle which shares τ with ν). However, the induced total bundles $\pi_{\Delta,1}$ and $\pi_{\Delta,2}$ are inequivalent since their Euler numbers differ by 1.

Notice that the parallel transport maps PT_1 and PT_2 agree when evaluated on any path contained in the 1-skeleton of $Sd(\Delta)$. Thus, this discrete set of parallel transport evaluations is not capable of detecting the difference between PT_1 and PT_2 . It is clear that given any other choice of discrete set of paths, with an appropriate choice of s we can fabricate a parallel transport map PT_2 which agrees with PT_1 when restricted to the chosen paths while inducing an inequivalent bundle.

The gluing between π_ν and π_τ is a change of trivialization of π_τ , from the one constructed using PT^τ to the one constructed using PT^ν . As any other change of trivialization it is described by an assignment $T_{\nu,\tau} : \tau \rightarrow U(1)$. The gluing assignment determined by PT_1 is $T_{\nu,\tau}(x) = \text{id}$ for all $x \in \tau$, while the gluing determined by PT_2 (which obeys compatibility conditions at $\partial\tau$) winds around $U(1)$ once. For a study of the abelian case see [2].

Now we describe (up to scale Δ) $[\{\pi_\nu \rightarrow_A \pi_\tau\}_{\tau \subset \partial\nu} \text{ (mod. equivalence)}, A \in \mathcal{A}/\mathcal{G}_{\pi_\nu}]$ in terms of parallel transport maps. The macroscopic data needed for such description is $\text{Data}_\Delta(PT^\nu) = (PT_\Delta^\nu, W^\nu)$ where

$$PT_\Delta^\nu : \mathcal{P}_\Delta(\nu) \rightarrow G$$

is the restriction of PT^ν to the discrete semigroup of paths $\mathcal{P}_\Delta(\nu) \subset \mathcal{P}(\nu)$ consisting of paths that fit in the 1-skeleton of $Sd(\nu)$. And,

$$W^\nu \in \{\text{Homotopy class of map } T_{\nu,\tau} : \tau \rightarrow G, \text{ with } T|_{\partial\tau} \text{ fixed}\}_{\tau \subset \nu},$$

where the gluing map $T_{\nu,\tau}$ is constructed from PT^ν following a variation of Barrett's construction [1]. The result is the following:

Theorem 2 (Faithful coarse graining) *The data $\text{Data}_\Delta(PT^\nu)$ characterizes*

- the gluing maps $\{\pi_\tau \rightarrow_{PT^\nu} \pi_\nu\}_{\tau \subset \partial\nu}$ up to equivalence and
- the connection modulo gauge $A \in \mathcal{A}/\mathcal{G}_{\pi_\nu}$ up to a microscopical deformation;

where $A, A' \in \mathcal{A}/\mathcal{G}_{\pi_\nu}$ are said to agree up to a microscopical deformation if they can be deformed to the same (singular) semigroup homomorphism by a homotopy of semigroup homomorphisms which preserves $\text{Data}_\Delta(PT^\nu)$.

In the cases relevant for two-dimensional and four-dimensional euclidean gravity ($[\dim M = 2, G = U(1)]$ and $[\dim M = 4, G = SU(2)]$ respectively) W^ν is characterized by a set of integers, one per codimension one face of ν . For many other cases, like the one relevant for three-dimensional euclidean gravity ($[\dim M = 3, G = SU(2)]$), W^ν is trivial.

This result suggests that if the current spin foam models for two-dimensional and four-dimensional gravity are modified to include all the relevant macroscopic degrees of freedom –the complete collection of macroscopic variables necessary to ensure faithful coarse graining–, then they could provide appropriate effective theories at a given scale.

A spin foam model for two-dimensional gravity which incorporates extra data regarding the bundle structure was given in [3]. In the same reference a naive spin foam model which ignored the bundle data was shown to yield unphysical results. An explicit relation between the work presented here and [3] will be presented elsewhere.

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3. References

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